

$$\vec{u} = u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k} \quad \mathbb{R}^3$$

$$\vec{v} = v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k}$$

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3 + \dots + u_n v_n$$

$$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\cdot: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$$

1) Symmetry or commutativity: $u \cdot v = v \cdot u$

2) Bilinearity:

$$T(\alpha x + \beta y, z) = \alpha T(x, z) + \beta T(y, z)$$

$$T(x, \alpha y + \beta z) = \alpha T(x, y) + \beta T(x, z)$$

3) Non-negativity: $u \cdot u \geq 0$ & $u \cdot u = 0$ if $u = 0$

$$u \cdot u = u_1^2 + u_2^2 + \dots + u_n^2 \quad (=0)$$

Such f^n that satisfy these 3 properties are called Inner product & the space $V \times V$ ($\mathbb{R}^n \times \mathbb{R}^n$) is called I.P.S.

$$u \cdot u = \|u\|^2 \quad \text{Norm of } u.$$

$$\|u\| = 1 \quad u \text{ is unit vector.}$$

Eg: $(1, 0, 0)$ $\|u\|^2 = 1^2 + 0^2 + 0^2 = 1$

$$\cos \theta = \frac{u \cdot v}{\sqrt{u_1^2 + u_2^2 + u_3^2} \sqrt{v_1^2 + v_2^2 + v_3^2}} \quad \mathbb{R}^3$$

$$\cos \theta = \frac{u \cdot v}{\|u\| \|v\|}$$

$$\|u\| = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2}$$

$$u = u_n$$

$$\cos \theta = 0$$

$$u \cdot v = 0$$

$$\theta = 90^\circ$$

$$\cos \theta = 0$$

$$u \cdot v = 0$$

$u \perp v$ or u is orthogonal to v .

eg: $\left\{ \underset{u}{(1,2)}, \underset{v}{(2,-1)} \right\}$ $u \cdot v = 1 \cdot 2 - 2 \cdot 1 = 0$

def: If in the set $\{v_1, v_2, \dots, v_n\}$ each $v_i \perp v_j$ $i \neq j$ then set is orthogonal set.

eg: $\left\{ \underset{v_1}{(2,1,-1)}, \underset{v_2}{(0,1,1)}, \underset{v_3}{(1,-1,1)} \right\}$

$$v_1 \cdot v_2 = 0 + 1 - 1 = 0 \quad v_2 \cdot v_3 = 0 - 1 + 1 = 0$$

$$v_1 \cdot v_3 = 2 - 1 - 1 = 0$$

Further if $\|v_i\| = 1 \quad \forall i$ then the set is called orthonormal set.

eg: $\left\{ \underset{v_1}{(1,0,0)}, \underset{v_2}{(0,1,0)}, \underset{v_3}{(0,0,1)} \right\}$

$$v_1 \cdot v_2 = 0 \quad v_2 \cdot v_3 = 0 \quad v_1 \cdot v_3 = 0$$

$$\|v_1\| = 1 \quad \|v_2\| = 1 \quad \|v_3\| = 1$$

Example of an orthonormal set

def: Basis of V which is orthogonal / orthonormal are called orthogonal / orthonormal basis.

Gram Schmidt orthogonalization process.

Given $\{x_1, x_2, \dots, x_n\}$

orthonormal $\{v_1, v_2, \dots, v_n\}$

Orthogonal $\{v_1, v_2, \dots, v_n\}$

Start: $v_1 = x_1$

$$v_2 = x_2 - \frac{(v_1 \cdot x_2) v_1}{\|v_1\|^2}$$

$$v_3 = x_3 - \frac{(v_1 \cdot x_3) v_1}{\|v_1\|^2} - \frac{(v_2 \cdot x_3) v_2}{\|v_2\|^2}$$

$$v_4 = x_4 - \frac{(v_1 \cdot x_4) v_1}{\|v_1\|^2} - \frac{(v_2 \cdot x_4) v_2}{\|v_2\|^2} - \frac{(v_3 \cdot x_4) v_3}{\|v_3\|^2}$$

Ex: $\left\{ \underset{x_1}{(1, 0, 1)}, \underset{x_2}{(1, 0, -1)}, \underset{x_3}{(0, 3, 4)} \right\}$

$$v_1 = (1, 0, 1)$$

$$v_2 = x_2 - \frac{(v_1 \cdot x_2) v_1}{\|v_1\|^2}$$

$$= (1, 0, -1) - 0 = (1, 0, -1)$$

$$v_3 = x_3 - \frac{(v_1 \cdot x_3) v_1}{\|v_1\|^2} - \frac{(v_2 \cdot x_3) v_2}{\|v_2\|^2}$$

$$= (0, 3, 4) - \frac{4}{2} (1, 0, 1) + \frac{4}{2} (1, 0, -1)$$

$$= (0, 3, 4) - (2, 0, 2) + (2, 0, -2)$$

$$= (0, 3, 0)$$

If $\{v_1, v_2, \dots, v_n\}$ is orthogonal then

$\{v_1, v_2, \dots, v_n\}$ is orthonormal

Let $\{v_1, v_2, \dots, v_n\}$ is orthonormal
 $\left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \dots, \frac{v_n}{\|v_n\|} \right\}$ is orthonormal

orthonormal $\left\{ \frac{1}{\sqrt{2}} (1, 0, 1), \frac{1}{\sqrt{2}} (1, 0, -1), \frac{1}{3} (0, 3, 0) \right\}$
 $= \left\{ \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right), (0, 1, 0) \right\}$

Eg: $\left\{ \underset{x_1}{(1, 1, 1, 1)}, \underset{x_2}{(1, 1, -1, -1)}, \underset{x_3}{(0, -1, 2, 1)} \right\}$

$v_1 = x_1 = (1, 1, 1, 1)$

$v_2 = x_2 - \frac{(v_1 \cdot x_2) v_1}{\|v_1\|^2} = (1, 1, -1, -1) - 0 = (1, 1, -1, -1)$

$v_3 = x_3 - \frac{(v_1 \cdot x_3) v_1}{\|v_1\|^2} - \frac{(v_2 \cdot x_3) v_2}{\|v_2\|^2}$

$= (0, -1, 2, 1) - \frac{2}{4} (1, 1, 1, 1) + \frac{4}{4} (1, 1, -1, -1)$

$= (0, -1, 2, 1) - \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) + (1, 1, -1, -1)$

$= \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right)$

Eg: $\left\{ (2, 0, 1), (3, -1, 5), (0, 4, 2) \right\}$

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Ans $\left\{ \frac{1}{\sqrt{5}} (2, 0, 1), \frac{1}{\sqrt{270}} (-7, -5, 14), \frac{1}{\sqrt{356}} (-1, 7, 2) \right\}$